A Numerical Investigation of the Multiply-and-Conquer Game

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The game "multiply and conquer" is a two-person zero-sum guessing game. Let $\mathbb{Z}_+$ be the set of strictly positive integers. Two players, $i = 1, 2$, simultaneously choose $x_i \in \mathbb{Z}_+$. The payoff to player 1 is given by

$$u_1(x_1, x_2) = \begin{cases} +x_2 & x_1 > x_2 \wedge \frac{x_1}{x_2} \in \mathbb{Z}_+ \\ +x_2 & x_2 > x_1 \wedge \frac{x_2}{x_1} \notin \mathbb{Z}_+ \\ -x_1 & x_1 > x_2 \wedge \frac{x_1}{x_2} \notin \mathbb{Z}_+ \\ -x_1 & x_2 > x_1 \wedge \frac{x_2}{x_1} \in \mathbb{Z}_+ \\ 0 & x_1 = x_2. \end{cases}$$

The motivation for the name of the game can be seen by inspection of the payoffs. The player with the larger integer wins if and only if his choice is an exact multiple of the choice of the other player. This game features the type of counterbalancing incentives that result in equilibria which involve randomization. On the one hand, choosing a large number is attractive, because it increases the chances of winning by being a multiple of the others player's choice; hence, the name of the game. On the other hand, choosing a large number is risky, in that if the other player chooses a small number that isn't a factor, the larger choice loses a large amount.

Open Question. Does the multiply and conquer game have a value?

The answer to the question is not clear. Multiply and conquer has an unbounded strategy space as well as an unbounded payoff function. Therefore, standard existence results for a value do not apply. However, if the game is arbitrarily restricted to having some maximum feasible strategy, then the game does have a value by the well-known result of VON NEUMANN [3]. Furthermore, the strategy attaining that value can be computed using linear programming (DANTZIG [1]). Denote the multiply-and-conquer game restricted to the strategy set $\{2, 3, \ldots, B\}$ by $G(B)$. It is evident by symmetry that the value of $G(B)$ is zero, and the value of the unrestricted game $G(\infty)$ must also be zero, if the value exists.

Because linear programming can be used to solve games $G(B)$, this paper investigates the equilibria of games $G(B)$ numerically. This turns out to be feasible for large values of B because the structure of the set of equilibria of $G(B)$ viewed as a function of B is simple. The equilibrium is unique for all values of B investigated, and that unique equilibrium has a support which is extremely sparse relative to B. Furthermore, as a function of B, the equilibrium changes infrequently. These properties can be understood intuitively as consequences of the number-theoretic structure of the game.

Throughout the following discussion, π_k is the equilibrium probability choice k is made. Explicitly manipulating the expressions for the expected payoff of a strategy is in general difficult due to the structure of the contingencies of winning and losing. However, for strategy $k = 2$, it is possible to write down a relationship that π must satisfy if $k = 2$ is played with positive probability. Assuming that the value of the game is zero, as it is in $G(B)$ for any $B < \infty$ and must be if $G(\infty)$ has a value, π must satisfy

$$\sum_{j=1}^{\infty}(2j+1)\pi_{2j+1} = \sum_{j=2}^{\infty} 2\pi_{2j}. \tag{1}$$

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*Preliminary and incomplete.

The left side of equation (1) expresses the gains from the contingencies where the opponent plays a choice which is not a multiple of 2, i.e., is odd. The right side calculates the losses from the contingencies where the opponent plays a multiple of 2, i.e., an even number. This equation can be rearranged to give the following interpretation,

$$2P(x_j \text{ even} \wedge x > 2) = P(x_j \text{ odd})E(x_j | x_j \text{ odd}). \tag{2}$$

Therefore, the right side of (2) is bounded above by 2. Manipulating further gives

$$2\pi_2 = 2P(x_j \text{ even}) - P(x_j \text{ odd})E(x_j | x_j \text{ odd}).$$

This relationship suggests that odd numbers will be used with small probability in equilibrium, and/or almost all the probability placed on odd numbers will be on small odd numbers. In particular, since all primes greater than 2 are odd, large prime numbers will be played very rarely.

Table 1 summarizes the key quantitative and qualitative features of the equilibria of $G(B)$ as a function of B, computed using the software package Gambit [2]. For all values of B, much of the probability weight, approximately three-fourths, is placed on the choices 2, 3, and 4. In addition, as expected, odd numbers are chosen relatively infrequently, and most of the probability weight placed on odd numbers is on the choice 3. The equilibria have sparse support. The columns Enter and Leave reflect the strategies which come into or depart the support at each critical value of B shown. The Enter column omits the value of B, which also enters the support; for instance, in the row where 240 enters the support, so do 18, 20, and 36. When the upper bound $B = 1,000,000$, only 23 choices are played with positive probability in equilibrium.

The behavior of the equilibrium as B changes can be understood through the following intuition. Absent the constraint B on the strategy set, for any mixed strategy profile that has a finite support, there is a number which can be played which would assure the opponent that he would "win" with probability one, and would earn a payoff equal to the expected value of the number chosen under that mixed strategy profile; this number is simply the product of all strategies played with positive probability. In equilibria of the restricted games $G(B)$, that strategy is not feasible. Most large numbers are not attractive as plays in the restricted game, since they would not multiply any numbers being played, or, if they did, would not win often enough. Remember that when a player does not win, he loses an amount equal to his choice, so losses are costly when large numbers are chosen. As B is allowed to increase, eventually a number which is a composite of choices played with high enough probability becomes feasible. For instance, consider the set of equilibria when $B \in 5040, 5041, \ldots, 10079, 10080$. As the numbers between 5041 and 10079 are added to the feasible set, they do not change the equilibrium, because they are not best replies to the equilibrium of $G(5040)$. However, 10080 is a strict best reply to that equilibrium, so when 10080 becomes feasible, it enters the equilibrium support. In "response," then, 11 also enters the support. In this case, 11 is the smallest prime number which was not yet being played with positive probability. That makes it attractive, because 11 wins over 10080, so the probability 10080 can be played in equilibrium must be small. Of course, other numbers which don't divide 10080 would also be defenses against 10080. Because 11 is prime, though, it does not multiply any smaller number by definition; therefore, defenses against the choice of 10080 should be the smallest number possible, because 11 will lose with high probability.

The equilibria presented in Table 1 appear to be converging. The last column, Dist, measures the distance the equilibrium changes at that value of B, as measured by the max-norm distance $d(\pi, \pi') = max_k |\pi_k - \pi'_k|$. The probabilities assigned to strategies do not change much as new strategies become feasible, nor are the strategies which enter the support played with large probability. However, this does not mean that this sequence is a sequence of ε-equilibria of the unrestricted game $G(\infty)$ for decreasing ε. Recall that by playing the product of all strategies in the support, a player could ensure victory with probability one, and obtain a payoff of $E(x)$. This $E(x)$ is increasing as B increases. However, because the equilibria change so little as new strategies are increased, it does show that for any equilibrium π^B of game $G(B)$, there exists a nearby strategy profile π' that makes deviation to any higher choice unprofitable.

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B	π_2	π_3	π_4	$P(\text{even})$	$E(x\|\text{even})$	$E(x\|\text{odd})$	$E(x)$	Enter	Leave	Dist
4	.4444	.2222	.3333	.7778	2.86	3.00	2.89	2,3,4		
8	.4304	.2278	.2911	.7722	3.08	3.00	3.06	6		.0422
12	.3190	.1637	.1115	.7586	5.34	3.64	4.93	5		.1797
24	.3217	.2325	.1991	.7487	7.81	3.40	6.70	9,10,16,18	8	.1264
48	.3273	.2269	.1834	.7475	7.91	3.33	6.76	8	16,18	.0937
240	.3460	.2253	.2069	.7557	8.84	3.35	7.50	18,20,36		.0446
720	.3592	.2039	.2048	.7673	9.12	3.51	7.81	7		.0214
5040	.3600	.2024	.2054	.7681	10.49	3.52	8.87	96		.0025
10080	.3586	.2040	.2052	.7675	15.06	3.52	12.38	11		.0022
110880	.3587	.2039	.2052	.7676	18.16	3.52	14.76	13		.0001

Table 1: Summary of properties of equilibria of $G(B)$.

References

[1] Dantzig, G. B., 1951. A proof of the equivalence of the programming problem and game problem. In Koopmans, T. C. (editor) 1951, *Activity Analysis of Production and Allocation*, New York, 330-335.

[2] McKelvey, R. D., A. M. McLennan, and T. L. Turocy. Gambit: Software Tools for Game Theory. http://gambit.sourceforge.net.

[3] von Neumann, J., 1928. Zur Theorie der Gesellschaftspiele, *Mathematische Annalen* 100(1): 295–320.