# Auction Choice for Ambiguity-Averse Sellers Facing Strategic Uncertainty

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March 26, 2007

**Abstract**

The robustness of the Bayes-Nash equilibrium prediction for seller revenue in auctions is investigated. In a framework of interdependent valuations generated from independent signals, seller expected revenue may fall well below the equilibrium prediction, even though the individual payoff consequences of suboptimal bidding may be small for each individual bidder. This possibility would be relevant to a seller who models strategic uncertainty as ambiguity, and who is ambiguity-averse in the sense of Gilboa and Schmeidler. It is shown that the second-price auction is more exposed than the first-price auction to lost revenue from the introduction of bidder behavior with small payoff errors.

**JEL classification:** D44, D81.

**Keywords:** auctions, strategic uncertainty, ambiguity aversion, robustness

## 1 Introduction

Auctions play a multitude of roles within economics and related fields. In deference to their long and broad use as a method of selling or purchasing goods, auction games are presented as one theoretical framework for price formation. This theoretical study, in turn, has begotten the design of custom market mechanisms, which may be auctions in their own right, or which may incorporate an auction as a component. Finally, auctions represent a common ground on which the empirical study of natural markets, the experimental study of auction games in the laboratory and the field all appear.

A workhorse solution concept in auctions in which bidders have private information is Bayes-Nash equilibrium. The seminal paper of Milgrom and Weber (1982) characterized the unique symmetric Bayes-Nash equilibrium in first-price and second-price auctions of a single, indivisible object with bidders with symmetric, affiliated values. This paper explores implications of the relaxation of the equilibrium assumption in a class of auction environments with interdependent values within the Milgrom and Weber framework. These environments have the property that, with no reserve price or positive minimum bid, the first-price and second-price auctions give the

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same expected revenue. It is shown that, for any given amount $\varepsilon$ of foregone expected payoffs to the bidders, the second-price auction is more vulnerable to revenue loss compared to the first-price auction. Furthermore, even bidding behaviors which result in only $\varepsilon$ foregone expected payoffs to bidders may result in much more than $\varepsilon$ lost profits to the seller.

With the removal of the equilibrium assumption, a seller now must contemplate possible scenarios for bidder behavior. This strategic uncertainty is modeled by supposing the seller may entertain many possible bidding scenarios which are consistent with bidders foregoing less than $\varepsilon$ in expected payoffs. That is, bidders are assumed to bid according to an $\varepsilon$-equilibrium, for $\varepsilon > 0$. Thus strategic uncertainty is treated as a source of ambiguity. A notion of robustness of an auction mechanism is then captured by considering the maxmin expected utility (MMEU) framework of Gilboa and Schmeidler (1989). The seller is interested in knowing, for each mechanism under consideration, the minimum expected revenue consistent with $\varepsilon$-equilibrium. Then, the ambiguity-averse seller would prefer to choose the mechanism with the largest minimum revenue among $\varepsilon$-equilibria.

The analysis complements an emerging interest in the literature in robustness of mechanisms, in particular to ambiguity. In applications to date, the assumption of noncooperative equilibrium has been maintained, while introducing ambiguity in the distribution of types, or of players' beliefs about the distribution of types. Implications of ambiguity in the value distribution have been investigated theoretically by Salo and Weber (1995) within a Choquet expected utility framework and Lo (1998) using the Gilboa-Schmeidler approach. Chen et al. (2007) show experimentally that ambiguity in the distribution of values results in lower bids in the first-price auction. Bose et al. (2006) consider ambiguity about the distribution of values for both the buyers and the seller, and characterize optimal mechanisms.

The maxmin criterion in Gilboa-Schmeidler can be viewed as a pessimistic attitude, and need not be generally applicable to all decision-makers. On the other hand, using maxmin in the presence of ambiguity carries with it a sense of robustness in the worst-case scenario. For example, Chung and Ely (2007) adopt a maxmin criterion in a justification for dominant strategy mechanisms. They show that, under a sufficient condition on the distribution of bidders' valuations, there exists some distribution of bidders' beliefs such that the best dominant-strategy mechanism will perform better than any detail-free mechanism. Their mechanism designer faces ambiguity in terms of bidders' beliefs about the distribution of other bidders' values, and considers the worst-case scenario.

An alternative motivation for focusing on bidding less than the Bayes-Nash equilibrium can be found in the (logit) quantal response equilibrium model of McKelvey and Palfrey (1995). Quantal response predicts underbidding for risk-neutral bidders in first-price uniform private values auctions. This is an implication of the observation of Friedman (1992) that the loss function from nonoptimal bidding in the first-price auction is asymmetric, with underbidding being less costly. Goeree at al. (2002) note that logit QRE predicts underbidding in their "low values" treatment, which is a discretized uniform private values auction; this prediction is independent of the coarseness of the discretization.

The paper is organized as follows. Section 2 describes the auction environment, which features a model of interdependent values from independent signals, indexed by one parameter. Section 3 characterizes the bidding profiles which minimize the seller's expected revenue, the quantity of interest to the ambiguity-averse seller, given that bidders make payoff errors no greater than some specified level $\varepsilon$. The argument is based in part on the Revelation Principle idea of self-revelation, except, instead, the characterization identifies what type the bidder would prefer to emulate given the out-of-equilibrium play. Section 4 numerically investigates the properties of the revenue-minimizing profiles, viewed as a function of $\varepsilon$. Section 5 concludes with discussion and directions for future inquiry. Where the proofs of results involve longer calculations, they are postponed to appendices to maintain the flow of the exposition.

## 2 Model

The auction environment fits the general symmetric affiliated values model of Milgrom and Weber (1982). There are $N$ risk-neutral bidders. Each bidder $i = 1, \ldots, N$ receives a private signal $x_i$, which is drawn from the uniform distribution on $[0, 1]$ and observed by bidder $i$ alone. The signals of the bidders are realized independently.

Conditional on the realization of the signals, the value of the object to bidder $i$ is given by

$$v_i = \beta x_i + (1 - \beta) \max_{j \neq i} x_j. \tag{1}$$

The parameter $\beta \in [0, 1]$ is an index of the extent to which there is interdependence of values among the bidders. When $\beta = 1$, this reduces to the independent private values model. In the case $\beta = \frac{1}{2}$ and $N = 2$, the model becomes the "wallet game." This value structure arises, for example, if bidders expect that there is some exogenous probability that they would need to re-sell the object at a later date; in such a case, finding out that there is another bidder with a high value makes the object more attractive. Alternatively, if the object for sale is a "prestige" good, ownership of the object is more attractive if someone else also places a high value on it.

For the purposes of this paper, this valuation model has several convenient features. The symmetric Bayes-Nash equilibria in both the first-price and second-price auctions is independent of $\beta$, and therefore both yield the same expected revenue to the seller. This makes comparisons for different levels of the interdependency parameter $\beta$ more straightforward. In addition, the value structure, along with the assumption that signals are uniformly distributed, permits an exact characterization of the solutions to the problem (2) below. The key property this value structure satisfies is that

$$E\left[\left(\max_{j \neq i} x_j\right) | x_i \geq x_j \forall j\right] = \frac{N-1}{N} x_i$$

is linear and increasing in $x_i$. The method in the proof arguments apply equally to any value structure $v_i = \beta x_i + (1 - \beta) f(x_{-i})$ where $E\left[f(x_{-i}) | x_i \geq x_j \forall j\right]$ is linear and increasing in $x_i$.

Strategic uncertainty is introduced by supposing that the seller is not certain that bidding in the auction will in fact be according to the equilibrium. The seller instead entertains the possibility that bidders bid in such a way that they may forgo as much as $\varepsilon > 0$ in expected earnings in the auction. That is, the seller considers the possibility that the bidders will emply bid functions constituting an $\varepsilon$-equilibrium (see, e.g., Radner (1980)). The seller is interested in the robustness of the expected revenue of the mechanisms to $\varepsilon$-equilibria, and therefore adopts a maxmin expected utility criterion as in Gilboa and Schmeidler (1989).

Throughout, symmetric bidding profiles are used, with $b(\cdot)$ denoting the bid function common to all bidders. The best-reply bidding function to $b$ is denoted $b^\star(b)$. For the $k$th-price auction $(k = 1, 2)$, the revenue to the seller if all bidders bid according to $b$ is written $R_k(b)$. The expected payoff to a bidder from following bid function $\hat{b}(\cdot)$ when all other bidders follow $b(\cdot)$ is $u_k(\hat{b}; b)$. With this notation, the programming problem of interest to the ambiguity-averse seller for the $k$-th price auction is

$$\begin{gathered} \rho_k(\varepsilon) = \text{minimize}_b \, R_k(b) \\ \text{subject to}\, u_k(b^\star(b); b) - u_k(b; b) \leq \varepsilon. \end{gathered} \tag{2}$$

Since the seller uses the maxmin criterion, he chooses the auction

$$k^\star(\varepsilon) = \text{argmax}_{k\in\{1,2\}}\, \rho_k(\varepsilon).$$

The intention is to find solutions to (2) which are "close" to the unique symmetric Bayes-Nash equilibrium of these games. Therefore, attention is restricted to bid functions which are strictly increasing, except possibly for an interval over the lowest signals $[0, y]$ on which the bid function may be zero. Additionally, where the bid function is strictly increasing, it is assumed that it is twice continuously differentiable. With these assumptions, the calculus of variations may be employed to characterize solutions to problem (2), which takes the form of an isoperimetric constrained optimization problem.

Finally, some convenient notation is introducted. First, define $\Delta \equiv (1-\beta)\frac{N-1}{N}$; this quantity appears frequently in capturing the expected value of the highest signal received by another bidder. Where it is convenient to work in terms of inverse bid functions, $\lambda(\cdot)$ will denote the common inverse bid function, and $\lambda^\star(\lambda)$ the (inverse of the) best-reply to the inverse bid function $\lambda$. In the case of a bid function that is zero on some interval $[0, y]$, define $\lambda(0) = y$, with the understanding that bidders with all signals below $\lambda(0)$ also submit a bid of zero.

# 3 Characterization of minimum-revenue curves

This section turns to the characterization of the solutions of problem (2) for the first-price and second-price auctions.

## 3.1 Best-response functions and symmetric equilibria

From the results of Milgrom and Weber (1982), it can be directly computed that the unique symmetric Bayes-Nash equilibrium in the first-price auction is

$$b^{1,\text{eqm}}(x) = \frac{N-1}{N}x$$

and the unique symmetric Bayes-Nash equilibrium in the second-price auction is

$$b^{2,\text{eqm}}(x) = x.$$

That is, the Bayes-Nash equilibrium bidding profiles are independent of $\beta$; therefore, revenues are independent of $\beta$.

However, the best-reply behavior does depend on $\beta$. The analysis will consider the formulation of the best-reply bidding function for one bidder, given that the other $N-1$ bidders follow identical increasing and $C^1$ bid functions. Let $[b_L, b_H]$ be the range of bids which are submitted by the other bidders. The best-reply bidding function in this environment is also increasing; furthermore, recall that, conditional on a given signal $x$, the expected payoff to submitting a bid $b$ is quasiconcave in $b$. Thus, if the first-order condition for an interior extremum holds for some $b$ in $(b_L, b_H)$, it is the maximizer; this will be referred to as the best reply being "interior" in what follows. Otherwise, the best-reply bid for signal $x$ occurs either at $b_L$ or $b_H$.

First consider the first-price auction, and suppose that all other bidders adopt the same $C^1$ increasing inverse bidding function $\lambda(\cdot)$. Then, a bid of $b$ wins with probability $\lambda(b)^{N-1}$. A bidder who receives signal $x$ chooses his bid $b$ to solve the maximization problem

$$\max_b \left[\beta x + (1-\beta)\frac{N-1}{N}\lambda(b) - b\right]\lambda(b)^{N-1}.$$

The first-order condition for an interior maximizer implies that the optimal bid $b^\star(x)$ satisfies

$$\beta x + (1-\beta)\lambda(b^\star(x)) - b^\star(x) = \frac{1}{N-1}\frac{\lambda(b^\star(x))}{\lambda'(b^\star(x))} \tag{3}$$

if it is interior. Since the best reply is weakly increasing in $x$, if no $b^\star(x)$ satisfies (3), then $b^\star(x) = b(1)$.

In the second-price auction, a bidder who receives signal $x$ and assumes all other bidders adopt the same $C^1$ inverse bidding function $\lambda(b)$ chooses his bid to solve

$$\max_b \left[\beta x + (1-\beta)\frac{N-1}{N}\lambda(b)\right]\lambda(b)^{N-1} - \int_0^b t(N-1)\lambda^{N-2}(t)\lambda'(t)dt,$$

where the integral captures the expected payment conditional on being the highest bidder. The first-order condition for an interior maximum reduces to

$$\beta x + (1-\beta)\lambda(b^\star(x)) - b^\star(x) = 0. \tag{4}$$

Note that for $\beta < 1$, the best reply in the second-price auction depends on $\lambda(\cdot)$, and that therefore the second-price auction does not have a symmetric equilibrium in weakly dominant strategies.

### 3.2 Ray bidding functions

First, consider the special case in which bidders adopt strategies of the form $b(x) = \alpha x$. Following Selten and Buchta (1998), these strategies will be referred to as "ray bidding" strategies, and the slope $\alpha$ referred to as a bid factor.[1] In addition to serving as a useful intermediate step for subsequent arguments, ray bidding strategies generally organize the data from first-price uniform private values auctions; see for example Cox et al. (1988).

Under the assumption that ray bidding strategies are adopted, the revenue-minimizing solutions to (2) give the same revenue for any choice of $\varepsilon$ in the first-price and second-price auctions.

**Proposition 1** *The optimal value of program (2) is the same for the first-price and second-price auctions when bidders use ray bidding strategies.*

**Proof** Pick some bid factor $\alpha$ in the second-price auction. Then, the bid factor $\frac{N-1}{N}\alpha$ gives the same revenue in the first-price auction:

**Lemma 1** *For any bid factor $\alpha$,*

$$R_1\left(\frac{N-1}{N}\alpha\right) = R_2(\alpha).$$

[1] Since there is a one-to-one relationship between bid factors and bid functions in this section, bid functions are denoted by their corresponding bid factor.

In addition, an exact correspondence between the values of deviating to the best-reply can be established:

**Lemma 2** *For any bid factor $\alpha$,*

$$u_1\left(A_1^\star\left(\frac{N-1}{N}\alpha\right), \frac{N-1}{N}\alpha\right) = u_2\left(A_2^\star(\alpha), \alpha\right)$$

*where $A_k^\star(\alpha)$ denotes the best-reply bid function when all other bidders follow bid factor $\alpha$.*

Both Lemmas 1 and 2 are established by direct calculation in Appendix A.

Since both bidding functions are strictly increasing, the object is allocated efficiently, and so bidders receive the same expected gains. Since the bidding is symmetric and the revenues are the same, bidders pay the same amount in expectation. Therefore, the expected utility of bidders to bidding according to bid factor $\alpha$ in the second-price auction is the same as following bid factor $\frac{N-1}{N}\alpha$ in the first-price auction. Lemma 2 states that the payoff to deviating to the best reply is the same in both situations. Therefore, the value of $\varepsilon$ is the same in both cases.

So, there is a one-to-one mapping between bid factors in the second-price auction with those in the first-price auction, which mapping preserves both revenue and $\varepsilon$; therefore, the optimal values for problem (2) must be the same. □

The idea behind the proof of Proposition 1 is that a one-to-one mapping can be established between strategies in the first- and second-price auctions. Suppose that bidders follow the symmetric bidding profile $b(x) = \alpha x$ in the second-price auction. Then, the revenue and the value of $\varepsilon$ corresponding to that bidding profile in the second-price auction are identical to the revenue and value of $\varepsilon$ corresponding to the bidding profile $b(x) = \frac{N-1}{N}\alpha x$ in the first-price auction.[2].

## 3.3 General bidding functions

The analysis now turns to the general case of strictly increasing, differentiable bidding functions. Since the domain of optimization is over functions, use of the calculus of variations is indicated. The proof strategy for both the first-price and second-price auctions is to use the Euler-Lagrange equation arising from (2) to argue that the first derivative of the revenue-minimizing bid function is constant over specified domains.

The goal is to express the problem in the form

$$\begin{array}{l} \text{minimize}_b \int_y^z F(w; b; b')dw \\ \text{subject to} \int_y^z G(w; b; b')dw \leq K. \end{array}$$

where $F$ and $G$ are functionals and $K$ is a scalar constant. Importantly, the limits of integration in both the maximand and the constraint must be the same (the interval $[y, z]$ in the template above). Accomplishing this depends on expressing the problem (2) in advantageous forms adapted to the two auction mechanisms.

For the first-price auction, it is convenient to consider the best-reply problem in a framing inspired by the Revelation Principle of mechanism design. Instead of computing the bid that maximizes the expected profit of a bidder of private signal $x$, write the problem as determining the

[2] This scaling property under ray-bidding strategies extends to the $k$th-price auctions of Kagel and Levin (1993); see Turocy (2001)

type $y > x$ which bidder $x$ would like to emulate, i.e., to submit a bid that would tie with that type, if other bidders are following $b(\cdot)$.[3] That is, the best reply can be characterized as choosing the type $y$ to solve

$$\max_y (\beta x + \Delta y - b(y))\, y^{N-1}$$

The first-order condition the optimal choice of $y$ must satisfy is

$$\beta x + \Delta y - b(y) = \frac{1}{N-1} \left[ b'(y) - \Delta \right] y. \tag{5}$$

Note that if $b(0) = 0$ and $b'(0) > 0$ only type zero would choose to win with probability zero, so $x(0) = 0$. For every signal $y$ there exists some signal $x < y$ such that a bidder with signal $x$ would optimally choose to bid the same in best reply as the bidder with signal $y$ bids in the given profile $b(\cdot)$. If $b$ is $C^1$, then $x(y)$ is a continuous function in $y$.

For the first-price auction, the revenue-minimizing bidding behavior for any given $\varepsilon$ is bidding a constant fraction of the private signal, i.e., a ray bidding function.

**Proposition 2** *For the first-price auction, the solutions to (2) are of the form*

$$\lambda(b) = kb$$

*for some* $k > \frac{N}{N-1}$. *That is, the solutions are ray bidding functions.*

**Proof** Deferred to Appendix C. □

Turning to the second-price auction, begin by recalling that the best-reply (inverse) bidding function $\lambda^\star(b)$ in the second-price auction is given by

$$\beta \lambda^\star(b) + (1-\beta)\lambda(b) - b = 0.$$

This implies that, for $\beta < 1$, if a positive measure of bidders bids zero according to $\lambda$, i.e., if $\lambda(0) > 0$, then the best-reply bid for a bidder of type zero is strictly positive. The lowest bid submitted in the best-reply function, denoted $b_M$, is then characterized by

$$b_M = (1-\beta)\lambda(b_M). \tag{6}$$

It then follows that the revenue and value integrals in problem (2) can be decomposed into two intervals, the bids in $[0, b_M]$ and the bids in $[b_M, 1]$. In view of (6), so long as $b_M$ is held fixed, it can be shown that the integrals on $[b_M, 1]$ are independent of the shape of $\lambda(\cdot)$ on $[0, b_M]$; the calculus of variations can then be used to argue that $\lambda(\cdot)$ must have a constant slope on $[b_M, 1]$. The second step uses this fact to characterize $\lambda(\cdot)$ on $[0, b_M]$, holding fixed the boundary condition that $(1-\beta)\lambda(b_M) = b_M$, and again uses the calculus of variations to argue that $\lambda(\cdot)$ must have a constant slope on $[0, b_M]$.

This therefore reduces the problem (2) to a search over a two-dimensional parameter space, given by $y$, the highest-type bidder who submits a bid of zero under $\lambda(\cdot)$, and $b_M$, the solution to (6). In view of the fact that the slope of $\lambda(\cdot)$ is constant on $[0, b_M]$, it is equivalent, and computationally

[3] The Revelation Principle approach to mechanism design (see, for example, Myerson (1979)) basically says that any given type $x$ would not prefer to imitate any other type $y \neq x$. Since the problem considers out of equilibrium profiles, that is not true here; indeed, bidders do want to bid differently than the bid function indicates. What is done here is to compute the type $y$ that a bidder of type $x$ wants to imitate, given that other bidders follow $b(x)$.

more convenient, to instead parameterize the search using the slope $\alpha$ of $\lambda(\cdot)$ on $[0, b_M]$. Given $y$ and $\alpha$, the solution to (6) is determined, and the slope $\beta$ of $\lambda(\cdot)$ on $[b_M, 1]$ is pinned down by the constraint that $\lambda(1) = 1$.

**Proposition 3** *For the second-price auction, the solutions to (2) are of the form*

$$\lambda(b) = \begin{cases} [0, y] & \text{if } b = 0 \\ y + \alpha b & \text{if } b \in [0, b_M] \\ y + \alpha b_M + \gamma(b - b_M) & \text{if } b \in [b_M, 1], \end{cases}$$

*where* $b_M = \frac{(1-\beta)y}{1-\alpha+\alpha\beta}$ *and* $\gamma = \frac{1-y-\alpha b_M}{1-b_M}$.

**Proof** Deferred to Appendix D. Figure 1 illustrates the candidate solutions and their corresponding best-reply schematically. □

In view of Proposition 1, Propositions 2 and 3 imply this corollary:

**Corollary 1** *For any given level of* $\varepsilon$*, the second-price auction admits lower expected revenues from a profile consistent with* $\varepsilon$*-equilibrium.*

Intuitively, the result stems from the fact that the payoff consequences to bidders of bidding nonoptimally are stronger as their signal increases, since the likelihood they have the largest value, or close to the largest value, increases. Since the first-price auction determines revenue via the highest bid submitted, the revenue and bidder payoff consequences are more closely aligned. On the other hand, the second-price auction determines revenue via the second-highest bid submitted; with symmetric bidding profiles, this implies that a lower-signal bidder is more likely to determine price. Thus, the connection is weaker between the individual incentives to bid accurately and the revenue consequences of bidding behavior.

## 3.4 A non-robustness result

With these results, it is possible to address the question of how sensitive the Bayes-Nash equilibrium point predictions are to the introduction of bidding errors with small payoff effects. Proposition 4 characterizes the non-robustness of the point prediction.

**Proposition 4** *For both the first- and second-price auctions,* $\lim_{\varepsilon \to 0} \rho'_k(\varepsilon) = -\infty$. *That is, allowing mistakes with small payoff implications for the bidders may result in a disproportionately large change in seller revenue.*

**Proof** The result can be established by considering ray bidding strategies. In view of Proposition 1, it is enough to consider the first-price auction. Observe that, for ray bidding strategies, there is a one-to-one correspondence between levels of foregone payoffs $\varepsilon$ and bid factors $\alpha$.[4] As such, it is well-defined to write both $\alpha(\varepsilon)$ and its inverse $\varepsilon(\alpha)$, and $\rho_k(\varepsilon) = R_k(\alpha(\varepsilon))$ is an identity. Furthermore, $\alpha(\varepsilon)$ is a $C^1$ function. Therefore, $\rho'_k(\varepsilon) = R_k(\alpha)\alpha'(\varepsilon) = R_k(\alpha)/\varepsilon'(\alpha)$.

[4] This statement takes into account that the analysis considers revenue minimizing bidding. There is also a value of $\alpha$ greater than the equilibrium bid factor that gives the same foregone earnings $\varepsilon$. Clearly, this gives greater-than-equilibrium seller revenue. The $\alpha(\varepsilon)$ refers to the lower value of $\alpha$, since that is the revenue minimizing one.

The revenue to the seller if all bidders adopt the bid function $b(x) = \alpha x$ is

$$R_1(\alpha) = \int_0^1 \alpha x \cdot N x^{N-1} dx,$$

and so $\frac{dR_1}{d\alpha}(\alpha) = N \int_0^1 x^N dx = \frac{N}{N+1} > 0$.

Next, turn to the constraint in (2), which can be written

$$\varepsilon(\alpha) = u_1(\alpha^\star(\alpha); \alpha) - u_1(\alpha; \alpha).$$

Here, $\alpha^\star(\alpha)$ denotes the best reply to $\alpha$. The best-reply equation (3) implies that the best reply is linear in the signal $x$, up until the point at which the best-reply bid exactly matches the highest bid submitted. That is, if $\alpha^\star$ is the slope of this best reply, then the best reply function is of the form

$$b^\star(\alpha) = \begin{cases} \alpha^\star x & \text{for } 0 \leq x \leq \frac{\alpha}{\alpha^\star} \\ \alpha & \text{for } \frac{\alpha}{\alpha^\star} < x \leq 1. \end{cases}$$

Let $u_1^d$ denote the derivative of $u_1$ with respect to its $d$th argument, $d = 1, 2$. Then differentiating the constraint, and evaluating at the equilibrium bid factor $\alpha^{\text{eqm}}$ gives

$$\varepsilon'(\alpha^{\text{eqm}}) = u_1^1(\alpha^{\text{eqm}}; \alpha^{\text{eqm}}) \frac{d\alpha^\star}{d\alpha} + u_1^2(\alpha^{\text{eqm}}; \alpha^{\text{eqm}}) - u_1^1(\alpha^{\text{eqm}}; \alpha^{\text{eqm}}) - u_1^2(\alpha^{\text{eqm}}; \alpha^{\text{eqm}}). \tag{7}$$

Since $\alpha^{\text{eqm}}$ is the best response against $\alpha^{\text{eqm}}$, $u_1^1(\alpha^{\text{eqm}}; \alpha^{\text{eqm}}) = 0$, and so $\varepsilon'(\alpha^{\text{eqm}}) = 0$. The right side of (7) is continuous, so $\lim_{\alpha \to \alpha^{\text{eqm}}} \varepsilon'(\alpha) = 0$. Therefore,

$$\lim_{\varepsilon \to 0} \rho_1'(\varepsilon) = \lim_{\alpha \to \alpha^{\text{eqm}}} \frac{R_1(\alpha)}{\varepsilon'(\alpha)} = \lim_{\alpha \to \alpha^{\text{eqm}}} \frac{N/(N+1)}{\varepsilon'(\alpha)} = -\infty.$$

□

# 4 Numerical analysis of solutions

Turning to quantitative considerations, Figure 2 illustrates the minimum revenues from problem (2) as a function of $\varepsilon$, $\rho_k(\varepsilon)$, for the first-price and second-price auction in four environments. The environments differ in two dimensions. The two rows of panels present the curves corresponding to small ($N = 2$) versus large ($N = 25$) auction environments. The two colums of panels present the curves corresponding to environments which are mostly private value ($\beta = 0.9$) and which are more common value ($\beta = 0.5$). Consistent with the foregoing analysis, the curve for the first-price auction (represented as a dashed line in the figures) lies above that for the second-price auction. Table 1 gives the numerical values of $\rho_k(\varepsilon)$ for selected values of $\varepsilon$.

The calculations admit two quantitative observations.

1. Proposition 4 shows that the $\rho_k(\varepsilon)$ curves should become vertical as $\varepsilon \to 0$. The calculations characterize the behavior of $\rho_k(\varepsilon)$ as $\varepsilon$ increases away from zero. For both auction institutions, even with small levels of foregone gains $\varepsilon$ by the bidders, there exist bidding profiles that have disproportionately large differences in revenue relative to the risk-neutral Bayes-Nash equilibrium baseline. For example, consider the environment with $N = 2$ and $\beta = 0.9$. Here, The minimum

| | $N=2$, $\beta=0.9$ | | $N=2$, $\beta=0.5$ | | $N=25$, $\beta=0.9$ | | $N=25$, $\beta=0.5$ | |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon$ | first | second | first | second | first | second | first | second |
| 0 | .333 | .333 | .333 | .333 | .923 | .923 | .923 | .923 |
| .005 | .277 | .243 | .315 | .262 | .831 | .816 | .890 | .874 |
| .010 | .254 | .206 | .274 | .230 | .794 | .763 | .826 | .818 |
| .025 | .210 | .143 | .241 | .170 | .720 | .643 | .770 | .718 |
| .050 | .160 | .090 | .203 | .113 | .637 | .520 | .705 | .619 |

Table 1: Minimum revenues consistent with symmetric $\varepsilon$-equilibrium, $\rho_k(\varepsilon)$, for four auction environments.

revenue for $\varepsilon = .01$ in the first-price auction is approximately .254, compared to the symmetric Bayes-Nash equilibrium prediction of $\frac{1}{3}$. For the second-price auction, this minimum revenue is approximately .206. When $N = 2$, the expected bidder profits in equilibrium are $\frac{1}{6}$; thus, $\varepsilon = .01$ is an optimization error of about five percent of total earnings; revenues, however, decrease by about one-third.

2. The introduction of inderdependent values has a salutary effect on the robustness of the auctions, with a stronger effect on the first-price auction. For $N = 2$ and $\beta = 0.5$, the first-price auction has a minimum revenue at $\varepsilon = .01$ of approximately .274, and the second price a minimum revenue of approximately .230. Thus, the minimum revenue increases for both auctions relative to the $\beta = 0.9$ case.

In view of the intuition behind the qualitative result that the second-price auction has lower revenue minima, namely, that the connection is weaker in the second-price auction between the incentives of the price-setting bidder to bid accurately and the revenue, it might be expected that as the number of bidders $N$ increases, the quantitative disparity between the mechansisms would decrease. The calculations indicate that the quantitative disparity is still significant at $N = 25$, which, for many applications, would constitute a large auction.

# 5 Conclusion

The non-robustness result (Proposition 4) illustrates that the point predictions of Bayes-Nash equilibrium are very sensitive to the assumption of mutual optimization of behavior among bidders in first-price and second-price auctions. Nonoptimal bidding that has only small payoff consequences for bidders, relative to their best response, may have a much larger impact on realized revenues. The bidders' optimization failure – whether due to computational or cognitive limitations, a lack of incentive to optimize, or collusive behavior – may impose a significant externality on the seller.

The non-robustness result is especially relevant for a seller who is ambiguity averse in the sense of Gilboa and Schmeidler. A natural and tractable extension of the Gilboa-Schmeidler maxmin criterion is the $\alpha$-MEU model, which was axiomatized by Ghirardato et al. (2004). In this model, a seller would evaluate an auction by taking $\alpha$ times the minimum revenue from problem (2) plus $1 - \alpha$ times the maximum revenue, computed as in problem (2) but with maximization replacing minimization. When $\alpha = 1$, this reduces to Gilboa-Schmeidler. The methods developed in this paper apply equally to the maximization variant of (2), and therefore a version of Corollary 1 on which institution would be preferred as a function of $\alpha$ could be obtained.

As was noted, the uniform distribution of signals and the structure of the valuations as a function of signals made it possible to characterize the solution of the Euler-Lagrange equations for the environment considered. A more thorough analysis will likely require numerical approaches to characterizing the $\rho_k(\varepsilon)$ curves for other auction environments or institutions. The contribution of this paper toward those extensions is to provide exact results for one class of environments to serve as a benchmark for verifying the results of a numerical procedure. The methods used to characterize the solutions illustrate potentially useful techniques for convenient ways to represent the problem for numerical analysis.

A final question for further study involves the assumption that payoff magnitudes are all that matter in modeling uncertainty over strategic choices. In comparing $\rho_1(\varepsilon)$ to $\rho_2(\varepsilon)$, an assumption is made that the magnitude of foregone payoffs $\varepsilon$ depends only on the environment – in this model, the interdependency parameter $\beta$ – and not on the auction institution. Whether this is a plausible assumption is an empirical question. In the pure private values case of $\beta = 1$, laboratory studies have shown that an ascending-clock (English) implementation of the second-price auction makes the weakly dominant strategy of $b(x) = x$ transparent, and thus, effectively, $\varepsilon = 0$. (See, for example, the survey of Kagel (1995).) However, for $\beta < 1$, the second-price auction no longer has a dominant strategy.

The ray-bidding result in Proposition 1 notes that, in the class of environments studied, there is a scaling relationship between the first-price and second-price auction. For any given among of deviation in terms of bidding behavior, the payoff consequences are smaller in the second-price auction compared to the first-price auction; or, conversely, it takes a greater deviation in bids in the second-price auction to have the same payoff consequences. Thus, a strategy method experiment in the spirit of Selten and Buchta (1998) in which subjects must choose a ray-bidding strategy may help address whether behavior, at least in the laboratory, is driven primarily by payoff consequences, or by the framing of the strategy space and rules of the specific institution.

## Acknowledgements

This project originally formed the second chapter of the author's PhD dissertation. Thanks to the author's thesis advisor, Mark Satterthwaite, and committee members James Dana, Karl Schmedders, and Robert Weber for comments and advice on the early versions, as well as seminar participants at SUNY-Albany, Duke University, T. J. Watson Labs at IBM, London School of Economics, the University of Miami, The Ohio State University, the University of Texas, Texas A&M University, and the University of Wisconsin-Madison. The comments of Dan Levin, John Kagel, and the late Ray Battalio shaped the redevelopment of the project in its current form, and comments from the editor and two anonymous referees sharpened the presentation. Stefan Jacewitz provided helpful research assistance in completing the presentation of the results. The author retains property rights over all errors.

## A Proof of Lemmas 1 and 2

**Proof (Lemma 1)** Given an inverse bidding function $\lambda(\cdot)$, the probability a bidder submits a bid less than $b$ is $\lambda(b)$ for $b \in [\lambda(0), \lambda(1)]$. So, for the second-price auction, the expected revenue is determined by the distribution of the second-highest bid. With a bid factor of $\alpha$, the induced distribution of bids from a bidder is uniform on $[0, \alpha]$, so

$$\begin{aligned} R_2(\alpha) &= \int_0^\alpha bN(N-1)\lambda(b)^{N-2}(1-\lambda(b))\lambda'(b)db \\ &= N(N-1)\int_0^\alpha b\left(\frac{b}{\alpha}\right)^{N-2}\left(1-\frac{b}{\alpha}\right)\frac{1}{\alpha}db = \alpha\frac{N-1}{N+1}. \end{aligned}$$

For the first-price auction, the revenue is determined by the distribution of the highest bid, which is

$$R_1(\alpha) = \int_0^\alpha bN\lambda(b)^{N-1}\lambda'(b)db = N\int_0^\alpha b\left(\frac{b}{\alpha}\right)^{N-1}\frac{1}{\alpha}db = \alpha\frac{N}{N+1}.$$

Therefore, $R_1\left(\frac{N-1}{N}\alpha\right) = R_2(\alpha)$. □

**Proof (Lemma 2)** Begin by characterizing the best-reply bid functions. It will be argued that the best-reply functions are of the form

$$b_k^\star(x) = \begin{cases} A_k^\star(\alpha)x & \text{for } x \in [0, \alpha/A_k^\star(\alpha)] \\ \alpha & \text{for } x \in [\alpha/A_k^\star(\alpha), 1]. \end{cases}$$

That is, the best-reply is to bid a constant fraction of the signal, up until the value of the signal at which doing so results in winning the auction with probability one; for higher values of the signal, bid the maximum bid submitted by the opponents (i.e., $\alpha$), and win with probability one.

For the second-price auction, the best-reply bid, when interior, satisfies (4), so

$$\begin{aligned} \beta x + (1-\beta)\frac{b^\star(x)}{\alpha} - b^\star(x) &= 0 \\ b^\star(x) &= \frac{\alpha\beta}{\alpha+\beta-1}x \equiv A_2^\star(\alpha)x. \end{aligned}$$

For the first-price auction, the best-reply bid, when interior, satisfies (3), so

$$\begin{aligned} \beta x + (1-\beta)\frac{b^\star(x)}{\alpha} - b^\star(x) &= \frac{1}{N-1}b^\star(x) \\ b^\star(x) &= \frac{\alpha\beta(N-1)}{N(\alpha+\beta-1)+(1-\beta)} \equiv A_1^\star(\alpha)x. \end{aligned}$$

So it follows that

$$\begin{aligned} A_1^\star\left(\frac{N-1}{N}\alpha\right) &= \frac{\frac{N-1}{N}\alpha\beta(N-1)}{N\left(\frac{N-1}{N}\alpha+\beta-1\right)+(1-\beta)} \\ &= \frac{N-1}{N}\cdot\frac{\alpha\beta(N-1)}{(N-1)(\alpha+\beta-1)} = \frac{N-1}{N}A_2^\star(\alpha). \end{aligned}$$

Next, it is shown that the expected value of the object conditional on winning, when using the best reply strategy (which is denoted $V_k^\star(\alpha)$) satisfies

$$V_1^\star\left(\frac{N-1}{N}\alpha\right) = V_2^\star(\alpha).$$

To see this, note that for $k = 1, 2$,

$$\begin{aligned} V_k^\star(\alpha) &= \int_0^{\frac{\alpha}{A_k(\alpha)}} \left(\beta x + \Delta \frac{A_k(\alpha)}{\alpha} x\right) \left(\frac{A_k(\alpha)}{\alpha}\right)^{N-1} x^{N-1} dx + \int_{\frac{\alpha}{A_k(\alpha)}}^1 (\beta x + \Delta)\, dx \\ &= \left(\beta + \Delta \frac{A_k(\alpha)}{\alpha}\right) \left(\frac{A_k(\alpha)}{\alpha}\right)^2 \frac{1}{N+1} + \frac{\beta}{2}\, x^2\Big|_{\alpha/A_k(\alpha)}^1 + \Delta x\Big|_{\alpha/A_k(\alpha)}^1 . \end{aligned}$$

Note that this depends on $\alpha$ only through the ratio $A_k(\alpha)/\alpha$. The preceding argument implies

$$\frac{A_1^\star\left(\frac{N-1}{N}\alpha\right)}{\frac{N-1}{N}\alpha} = \frac{A_2^\star(\alpha)}{\alpha}$$

which, applied to the expressions for $V_1^\star\left(\frac{N-1}{N}\alpha\right)$ and $V_2^\star(\alpha)$, gives the result.

Finally, it is shown that the expected amounts paid when using the best-reply satisfy

$$P_1^\star\left(\frac{N-1}{N}\alpha\right) = P_2^\star(\alpha).$$

To establish this, proceed by direct calculation for both auction mechanisms, dividing the integrals into the regions on which the best reply wins with probability less than one, and on which it wins with probability one. For the second-price auction,

$$\begin{aligned} P_2^\star(\alpha) &= \int_0^{\frac{\alpha}{A_2^\star(\alpha)}} dx \left[\int_0^{A_2^\star(\alpha)x} (N-1) b \left(\frac{b}{\alpha}\right)^{N-2} \frac{1}{\alpha} db\right] \\ &\quad + \int_{\frac{\alpha}{A_2^\star(\alpha)}}^1 dx \left[\int_0^{\alpha} (N-1) b \left(\frac{b}{\alpha}\right)^{N-2} \frac{1}{\alpha} db\right] \\ &= \frac{N-1}{N}\left[\frac{\alpha^2}{A_2^\star(\alpha)} \frac{1}{N+1} + \alpha\left(1 - \frac{\alpha}{A_2^\star(\alpha)}\right)\right]. \end{aligned}$$

For the first-price auction

$$\begin{aligned} P_1^\star(\alpha) &= \int_0^{\frac{\alpha}{A_1^\star(\alpha)}} (A_1^\star(\alpha)x) \left(\frac{A_1^\star(\alpha)}{\alpha} x\right)^{N-1} dx + \int_{\frac{\alpha}{A_1^\star(\alpha)}}^1 \alpha dx \\ &= \frac{\alpha^2}{A_1^\star(\alpha)} \frac{1}{N+1} + \alpha\left(1 - \frac{\alpha}{A_1^\star(\alpha)}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} P_1^\star\left(\frac{N-1}{N}\alpha\right) &= \frac{\left(\frac{N-1}{N}\alpha\right)^2}{A_1^\star\left(\frac{N-1}{N}\alpha\right)} \cdot \frac{1}{N+1} + \frac{N-1}{N}\alpha\left(1 - \frac{\frac{N-1}{N}\alpha}{A_1^\star\left(\frac{N-1}{N}\alpha\right)}\right) \\ &= \frac{\left(\frac{N-1}{N}\alpha\right)^2}{\frac{N-1}{N} A_2^\star(\alpha)} \cdot \frac{1}{N+1} + \frac{N-1}{N}\alpha\left(1 - \frac{\frac{N-1}{N}\alpha}{\frac{N-1}{N} A_2^\star(\alpha)}\right) \\ &= \frac{N-1}{N}\left[\frac{\alpha^2}{A_2^\star(\alpha)} \cdot \frac{1}{N+1} + \alpha\left(1 - \frac{\alpha}{A_2^\star(\alpha)}\right)\right] \\ &= \frac{N-1}{N} P_2^\star(\alpha). \end{aligned}$$

Therefore, the payoff to deviating to the best reply in the second-price auction when other bidders bid using bid factor $\alpha$ is the same as the payoff to deviating to the best reply in the first-price auction when other bidders bid using bid factor $\frac{N-1}{N}\alpha$, which is the claim of the lemma. □

# B Note on use of calculus of variations

In Appendices C and D, the calculus of variations is used to characterize the shape of the bid functions that solve problem (2). This appendix covers notes on this process common to the two proofs.

The goal is to express the problem (2) for the first-price auction in the form

$$\begin{aligned}&\text{minimize}_{\phi} \int_y^z F(w;\phi;\phi')dw\\ &\text{subject to} \int_y^z G(w;\phi;\phi')dw \leq K.\end{aligned}$$

for some functionals $F$ and $G$ and some scalar constant $K$ that is independent of $\phi(\cdot)$. The Euler-Lagrange equation characterizing an extremum is given by

$$\frac{\partial}{\partial w}\left[\frac{\partial F}{\partial \phi'} - \mu\frac{\partial G}{\partial \phi'}\right] - \left[\frac{\partial F}{\partial \phi} - \mu\frac{\partial G}{\partial \phi}\right] = 0,$$

where $\mu$ is a scalar Lagrange multiplier. Since the functionals appear only through their derivatives, this implies that any additive terms in $F$ and $G$ that are independent of the variation of $\phi(\cdot)$ can be neglected, since they will not appear in the Euler-Lagrange equation.

In the calculations, when such additive constants appear, generally through the application of integration by parts, they will be shown only in the first equation, and will be dropped thereafter. At the points where such constants are "disappeared," the notation $=^{\star}$ will be used, which can be read as "equal up to an additive constant."

Finally, in both proofs, the proof strategy will include assuming certain boundary conditions on $\phi$ and $\phi'$ at the endpoints of integration. Thus, the calculus of variations arguments themselves characterize the function $\phi$ given those endpoints; in the case of both arguments, the argument will be that $\phi'$ is constant on the relevant domain $[y, z]$. To solve the original problem (2), then, one considers all possible choices of boundary conditions, applying the fact that $\phi'$ is constant on the domain, to compute the overall solution.

# C Proof of Proposition 2

### Some preliminaries

Recall that when the best reply bid is interior (that is, wins with probability strictly between zero and one), the best reply behavior for a signal $x$ can be characterized by

$$\beta x + \Delta y - b(y) = \frac{1}{N-1}\left[b'(y) - \Delta\right] y, \tag{8}$$

where $y$ is interpreted as the type of bidder a bidder with signal $x$ would choose to imitate, assuming bidders bid according to the bid function $b(\cdot)$. It will be convenient to work in terms of the function

$$\phi(y) = (N-1)^{-1}[b'(y) - \Delta].$$

For signals for which the best-reply behavior is interior, the solutions to (8) define a function $x(y)$. The signal for which a bidder with that signal would choose to imitate the highest type (and therefore win with probability one) is given by

$$\beta x(1) = \frac{1}{N-1}\left[b'(1) - \Delta\right] + b(1) - \Delta.$$

Note that $x(1)$ only depends on $b(\cdot)$ via $b(1)$ and $b'(1)$. Therefore, for the calculus of variations argument, consider functions $b(\cdot)$ satisfying $b(1) = c_0$ and $b'(1) = c_1$ for some fixed constants $c_0$ and $c_1$. Finally, the derivative of $x(y)$ is computed by differentiating (8):

$$\begin{aligned} \beta x'(y) + \Delta - b'(y) &= \phi(y) + \phi'(y)y \\ x'(y) &= \beta^{-1}\left[N\phi(y) + \phi'(y)y\right]. \end{aligned}$$

**Expected revenue to the seller**

The expected revenue to the seller can be expressed as

$$R_1(b) = \int_0^1 Nb(x)x^{N-1}dx.$$

Integrating by parts gives

$$R_1(b) = b(1) - \int_0^1 b'(x)x^N dx.$$

Neglecting the additive constant and rewriting the intergral in terms of $\phi(\cdot)$ gives

$$R_1(b) =^{\star} - \int_0^1 \left[(N-1)\phi(x) + \Delta\right] x^N dx \tag{9}$$

**Expected payoff to bidding according to the symmetric profile**

The payoff to bidding according to the symmetric profile $b(\cdot)$ is

$$\begin{aligned} u_1(b;b) &= \int_0^1 \left[\beta x + \Delta x - b(x)\right] x^{N-1} dx \\ &= (N+1)^{-1}(\beta + \Delta) - \int_0^1 b(x)x^{N-1}dx \\ &= (N+1)^{-1}(\beta + \Delta) - N^{-1}b(1) + N^{-1}\int_0^1 b'(x)x^N dx. \end{aligned}$$

Neglecting the additive constants and rewriting the intergral in terms of $\phi(\cdot)$ gives

$$u_1(b;b) =^{\star} N^{-1} \int_0^1 \left[(N-1)\phi(x) + \Delta\right] x^N dx. \tag{10}$$

**Expected payoff to following best reply**

The payoff to following the best reply can be written in two parts. For the types for whom the optimal bid is interior, write

$$\begin{aligned}\int_0^1 [\beta x(y) + \Delta y - b(y)]\, y^{N-1} x'(y) dy &= \int_0^1 \phi(y) y \times y^{N-1} \times \beta^{-1} [N\phi(y) + \phi'(y)y]\, dy \\ &= \beta^{-1} \int_0^1 \left[N\phi^2(y) y^N + \phi(y)\phi'(y) y^{N+1}\right] dy.\end{aligned}$$

For the types for which the optimal bid is $b(1)$, the expected payoff is

$$\int_{x(1)}^1 [\beta x + \Delta - b(1)]\, dx = \frac{\beta}{2}\left[1 - x(1)^2\right] + [\Delta - b(1)]\, [1 - x(1)]\, .$$

Since $b(1)$ and $b'(1)$ are taken as fixed, $x(1)$ is fixed, and this is a constant. So, neglecting the additive constants,

$$u_1(b^\star(b); b) =^\star \beta^{-1} \int_0^1 N\phi^2(x) x^N + \phi(x)\phi'(x) x^{N+1} \Big] dx \tag{11}$$

**Characterization of extremum**

Taking (9), (10), and (11) together, the problem is formulated as

$$\text{maximize}_\phi - \int_0^1 [(N-1)\phi(x) + \Delta]\, x^N dx$$

subject to the constraint

$$\beta^{-1} \int_0^1 N\phi^2(x) x^N + \phi(x)\phi'(x) x^{N+1} \Big] dx - N^{-1} \int_0^1 [(N-1)\phi(x) + \Delta]\, x^N dx \leq K.$$

For the Euler-Lagrange equation, compute

$$\begin{aligned}\frac{\partial F}{\partial \phi'} &= 0 \\ \frac{\partial G}{\partial \phi'} &= \beta^{-1}\phi(x) x^{N+1} \\ \frac{\partial}{\partial x}\left[\frac{\partial G}{\partial \phi'}\right] &= (N+1)\beta^{-1}\phi(x) x^N \\ \frac{\partial F}{\partial \phi} &= -(N-1) x^N \\ \frac{\partial G}{\partial \phi} &= 2\beta^{-1} N\phi(x) x^N + \beta^{-1}\phi'(x) x^{N+1} - \frac{N-1}{N} x^N.\end{aligned}$$

The Euler-Lagrange equation then evaluates to

$$-\mu(N+1)\beta^{-1}\phi(x) x^N + (N-1) x^N + 2\mu\beta^{-1} N\phi(x) x^N + \mu\beta^{-1}\phi'(x) x^{N+1} - \mu\frac{N-1}{N} x^N = 0.$$

Gathering terms and dividing through by $x^N$, this is of the form

$$\theta_1 \phi(x) + \theta_2 + \theta_3 \phi'(x)x = 0,$$

which is satisfied if and only if $\phi$ is constant. Therefore, $b'$ is constant, and so $b$ is linear. Since $b$ is linear, it follows that $b(1) = c_0 = c_1 = b'(1)$. Thus, determiing the solution to problem (2) for the first-price auction reduces to finding the bid factor $\alpha$ such that there would be an expected payoff gain of $\varepsilon$ to deviating from the corresponding ray bidding function to the best reply.

# D Proof of Proposition 3

**Some preliminaries**

Recall that the best-reply (inverse) bidding function in the second-price auction is given by

$$\beta\lambda^\star(b) + (1-\beta)\lambda(b) - b = 0.$$

This implies that, for $\beta < 1$, if a positive measure of bidders bids zero according to $\lambda$, i.e., if $\lambda(0) > 0$, then the best-reply bid for a bidder of type zero is strictly positive. The lowest bid submitted in the best-reply function is then characterized by

$$b_M = (1-\beta)\lambda(b_M).$$

Further, the analysis considers bid functions for which the best-reply function is such that no bidder with value less than one wins with probability one in the best-reply. Thus, the best-reply characterization implies that the maximum bid submitted is $b^{\max} = 1$.

The chararacterization of the solution of (2) will proceed by the calculus of variations in two steps. First, it will be assumed that $\lambda(\cdot)$ is fixed and given for all bids on $[0, b_M]$, where $b_M$ solves (6), and it will be shown that $\lambda(\cdot)$ on $[b_M, 1]$ has a constant slope. Then, it will be assumed that $\lambda(\cdot)$ is fixed and has constant slope on $[b_M, 1]$, and it will be shown that $\lambda(\cdot)$ has constant slope on $[0, b_M]$.

It will be convenient to define the function

$$\Lambda(b) = \int_0^b \lambda(w)^{N-1} dw.$$

for the application of the calculus of variations. Note that this definition implies

$$\Lambda'(b) = \lambda(b)^{N-1}$$

and

$$\lambda(b) = \Lambda'(b)^{\frac{1}{N-1}}.$$

## Step 1: Characterizing $\lambda(\cdot)$ on $[b_M, 1]$

This step begins by assuming that $\lambda(\cdot)$ is fixed and given on the interval $[0, b_M]$, and aims to characterize $\lambda(\cdot)$ on the interval $[b_M, 1]$. The quantities of interest in the problem are analyzed separately.

**Expected revenue to the seller**

For a given symmetric inverse bidding profile $\lambda(\cdot)$, the revenue to the seller in the second price auction can be written

$$R_2(\lambda) = \int_0^1 bN(N-1)\lambda(b)^{N-2}(1-\lambda(b))\lambda'(b)db.$$

Observe that the portion of the integral in (12) on $[0, b_M]$ is constant, as it only depends on $\lambda(\cdot)$ on that domain. Thus, analysis focuses on the integral in (12) where the lower bound of integration is changed to $b_M$.

Using integration by parts, this can be rewritten

$$\begin{aligned} R_2(\lambda) &= N\left[b\lambda(b)^{N-1}\big|_{b_M}^1 - \int_{b_M}^1 \lambda(b)^{N-1}db\right] + (N-1)\left[b\lambda(b)^N\big|_{b_M}^1 - \int_{b_M}^1 \lambda(b)^N db\right] \\ &=^\star \int_{b_M}^1 \left[N\lambda(b)^{N-1} - (N-1)\lambda(b)^N\right] db = \int_{b_M}^1 \left[(N-1)\Lambda'(b)^{\frac{N}{N-1}} - N\Lambda'(b)\right] db. \quad (12) \end{aligned}$$

**Expected value to bidder conditional on winning, given $\lambda(\cdot)$**

Since the bidding profiles being considered are symmetric, and since $\lambda(0)$ is fixed, the expected value to the bidder, conditional on winning, is independent of the shape of $\lambda(\cdot)$. If the bid function were strictly increasing, this value would be $\frac{1}{N+1}$; however, since $\lambda(0)$ may be strictly positive, there is a positive probability of ties, which occur when all bidders have value less than $\lambda(0)$; thus, allocation need not be efficient. In any case, since $\lambda(0)$ is taken as given, the variation of $\lambda(\cdot)$ does not change this quantity, and thus the expected value to the bidder conditional on winning may be neglected as a constant.

**Expected payment of bidder, given $\lambda(\cdot)$**

Observe that, since bidding profiles are symmetric, the amount paid in expectation by the bidder must be $N^{-1}R_2(\lambda)$.

**Expected value to bidder conditional on winning, under best reply**

This can be calculated directly, using the characterization (4) of the inverse best reply. Write $\delta = 1-\beta$, and note that only bids on the interval $[b_M, 1]$ are submitted in the best reply.

$$\begin{aligned} EV &= \int_{b_M}^1 [\beta\lambda^\star(b) + \Delta\lambda(b)]\lambda(b)^{N-1}\lambda'^\star(b)db \\ &= \beta^{-1}\int_{b_M}^1 \left[(b + (\Delta-\delta)\lambda(b))\lambda(b)^{N-1}(1-\delta\lambda'(b))\right] db \\ &= \beta^{-1}\int_{b_M}^1 \left[b\lambda(b)^{N-1} - \delta b\lambda(b)^{N-1}\lambda'(b) + (\Delta-\delta)\lambda(b)^N - (\Delta-\delta)\delta\lambda(b)^N\lambda'(b)\right] db \quad (13) \end{aligned}$$

**Expected payment of bidder, under best reply**

This can also be calculated directly, using (4).

$$\begin{aligned} EP &= \int_{b_M}^{1}\left[\int_0^b w(N-1)\lambda(w)^{N-2}\lambda'(w)dw\right]\beta^{-1}(1-\delta\lambda'(b))db \\ &= \beta^{-1}\int_{b_M}^{1}\left[w\lambda(w)^{N-1}\,|_0^b - \int_0^b \lambda(w)^{N-1}dw\right](1-\delta\lambda'(b))db \\ &= \beta^{-1}\int_{b_M}^{1}\left[b\lambda(b)^{N-1} - \delta b\lambda(b)^{N-1}\lambda'(b) - \Lambda(b) + \delta\lambda'(b)\Lambda(b)\right]db. \end{aligned} \quad (14)$$

The expected payoff of deviating to the best reply is just $EV - EP$. Note that the first two terms of (13) match the first two terms of (14). Thus,

$$EV - EP = \beta^{-1}\int_{b_M}^{1}\left[(\Delta-\delta)\lambda(b)^N - (\Delta-\delta)\delta\lambda(b)^N\lambda'(b) - \Lambda(b) + \delta\lambda'(b)\Lambda(b)\right]db \quad (15)$$

The first and third terms in the integrand of (15) can be written directly in terms of $\Lambda$ and $\Lambda'$. To express the others,

$$\int_{b_M}^{1}\lambda(b)^N\lambda'(b)db = \lambda(b)^{N-1}\Big|_{b_M}^{1} - \int_{b_M}^{1}\lambda(b)^N db =^\star -\int_{b_M}^{1}\Lambda'(b)^{\frac{N}{N-1}}db$$

and

$$\begin{aligned} \int_{b_M}^{1}\lambda'(b)\Lambda(b)db &= \lambda(b)\Lambda(b)\,|_{b_M}^{1} - \int_{b_M}^{1}\lambda(b)^N db \\ &= \lambda(1)\Lambda(1) - \lambda(b_M)\Lambda(b_M) - \int_{b_M}^{1}\lambda(b)^N db \\ &= \Lambda(1) - \lambda(b_M)\Lambda(b_M) - \int_{b_M}^{1}\lambda(b)^N db \\ &= \int_{b_M}^{1}(\lambda(b)^{N-1} - \lambda(b)^N)db + (1-\lambda(b_M))\int_0^{b_M}\lambda(b)^{N-1}db \\ &=^\star \int_{b_M}^{1}\left(\Lambda'(b) - \Lambda'(b)^{\frac{N}{N-1}}\right)db. \end{aligned}$$

The key step is to write out $\Lambda(1) = \int_0^1 \lambda(b)^{N-1}db$, and to decompose this integral into two integrals over $[0, b_M]$ and $[b_M, 1]$, respectively.

Therefore, expected earnings for playing the best-reply function $\lambda^\star(\cdot)$ can be written as

$$EV - EP =^\star \beta^{-1}\int_{b_M}^{1}\left[(\Delta-\delta)(1+\delta)\Lambda'(b)^{\frac{N}{N-1}} - \delta\Lambda'(b)^{\frac{N}{N-1}} + \delta\Lambda'(b) - \Lambda(b)\right]db, \quad (16)$$

up to an additive constant.

**Characterization of solution via the calculus of variations**

In view of (12) and (15), problem (2), under the assumption that the function $\lambda(\cdot)$ is fixed and given on $[0, b_M]$, can be re-expressed as

$$\begin{gathered}\text{minimize}_\Lambda \int_{b_M}^1 \left[\nu_1 \Lambda'(b)^{\frac{N}{N-1}} - \nu_2 \Lambda'(b)\right] db \\ \text{subject to} \int_{b_M}^1 \left\{\nu_3 \Lambda'(b)^{\frac{N}{N-1}} + \nu_4 \Lambda'(b) - \Lambda(b)\right\} db = Q,\end{gathered}$$

where $\nu_1$, $\nu_2$, $\nu_3$, $\nu_4$, and $Q$ are constants depending on $\beta$ and $N$. The Euler-Lagrange equation for this problem is

$$\mu + \frac{d}{db}\left[N\nu_1 \Lambda'(b)^{\frac{1}{N-1}} - \nu_2 - \mu\left(N\nu_3 \Lambda'(b)^{\frac{1}{N-1}} + \nu_4\right)\right] = 0,$$

where $\mu$ is the Lagrange multiplier on the constraint. Integrating gives

$$\mu(b - b_M) + \kappa_1 \Lambda'(b)^{\frac{1}{N-1}} + \kappa_2 = \kappa_0,$$

where $\kappa_0$ is the constant of integration and $\kappa_1$ and $\kappa_2$ are amalgams of $N$, $\delta$, and $\mu$. Recalling that $\lambda(b) = \Lambda'(b)^{\frac{1}{N-1}}$, solving this equation gives

$$\lambda(b) = \frac{\kappa_0 - \kappa_2 - \mu(b - b_M)}{\kappa_1}$$

and thus $\lambda'(b)$ is constant, as claimed.

## Step 2: Characterizing $\lambda(\cdot)$ on $[0, b_M]$

Next, consider all functions $\lambda(\cdot)$ such that $\lambda'(\cdot)$ is constant above the bid $b_M$ defined by (6).

**Expected revenue to the seller**

Following the same procedure as in Step 1, the portion of the revenue function which depends on the variation in $\lambda(\cdot)$ on $[0, b_M]$ is

$$R_2(\lambda) =^\star \int_0^{b_M} \left[(N-1)\Lambda'(b)^{\frac{N}{N-1}} - N\Lambda'(b)\right] db$$

**Expected value to bidder conditional on winning, given $\lambda(\cdot)$**

As in Step 1, since only symmetric bidding profiles are considered, this quantity does not depend on the variation in $\lambda(\cdot)$.

**Expected payment of bidder, given $\lambda(\cdot)$**

Because of symmetry, this is simply $N^{-1}R_2(\lambda)$.

**Expected value to bidder conditional on winning, under best reply**

Note that (4) implies that the best-reply inverse bidding function does not depend on the shape of $\lambda(\cdot)$ below $b_M$. Thus, the expected value conditional on winning under the best reply is constant with respect to the variation of $\lambda(\cdot)$ on $[0, b_M]$, so long as the value of $b_M$ solving (6) is unchanged.

**Expected payment of bidder, under best reply**

Note that the only terms in (14) which vary with respect to variation in $\lambda(\cdot)$ on $[0, b_M]$ are those that involve $\Lambda$. Thus, the variable portion is

$$\begin{aligned}\int_{b_M}^{1} [\delta\lambda'(b)\Lambda(b) - \Lambda(b)]\, db &= \int_{b_M}^{1} \Bigg[\left(\delta\lambda'(b)\int_0^{b_M} \lambda(w)^{N-1}dw\right) + \left(\delta\lambda'(b)\int_{b_M}^{b} \lambda(w)^{N-1}dw\right) \\ &\quad - \int_0^{b_M} \lambda(w)^{N-1}dw - \int_{b_M}^{b} \lambda(w)^{N-1}dw\Bigg]\, db \\ &=^{\star} \int_{b_M}^{1} \left(\delta\lambda'(b)\int_0^{b_M} \lambda(w)^{N-1}dw\right) - \int_0^{b_M} \lambda(w)^{N-1}dw \\ &= \int_0^{b_M} (1-b_M)\left(\delta C\lambda(w)^{N-1} - \lambda(w)^{N-1}\right) dw,\end{aligned}$$

where the exchanging of the order of integration in the last line is justified since $\lambda'(b) = C$ for some constant $C$ on $[b_M, 1]$. Thus, for the portion of the expected payment under the best reply that varies,

$$\int_{b_M}^{1} [\delta\lambda'(b)\Lambda(b) - \Lambda(b)]\, db =^{\star} M\int_0^{b_M} \lambda(b)^{N-1}db = M\int_0^{b_M} \Lambda'(b)db$$

where $M$ is a constant.

**Characterization of solution via the calculus of variations**

The foregoing has established that the solution to problem (2), under the assumption that the function $\lambda(\cdot)$ is fixed and given on $[b_M, 1]$ such that $\lambda'$ is constant, is the same as the solution to the problem

$$\begin{aligned}\text{minimize} \int_0^{b_M} \left[(N-1)\Lambda'(b)^{\frac{N}{N-1}} - N\Lambda'(b)\right] db \\ \text{subject to} \int_0^{b_M} \left\{M\Lambda(b) - \tfrac{N-1}{N}\Lambda'(b)^{\frac{N}{N-1}} + \Lambda'(b)\right\} db = Q,\end{aligned}$$

where $Q$ is a constant that incorporates both $\varepsilon$ and the various constants appearing in the foregoing derivations. The Euler-Lagrange equation for this problem is

$$\mu + \frac{d}{db}\left[N\Lambda'(b)^{\frac{1}{N-1}} - N - \mu\left(-\Lambda'(b)^{\frac{1}{N-1}} + 1\right)\right] = 0.$$

Following in parallel to Step 1, this implies that $\lambda'(\cdot)$ is constant on $[0, b_M]$.

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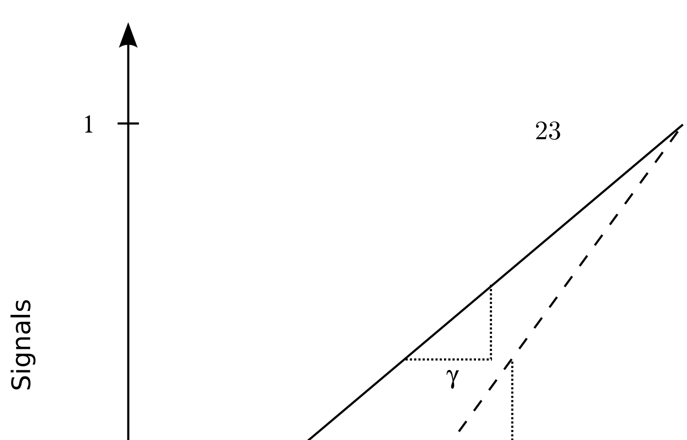
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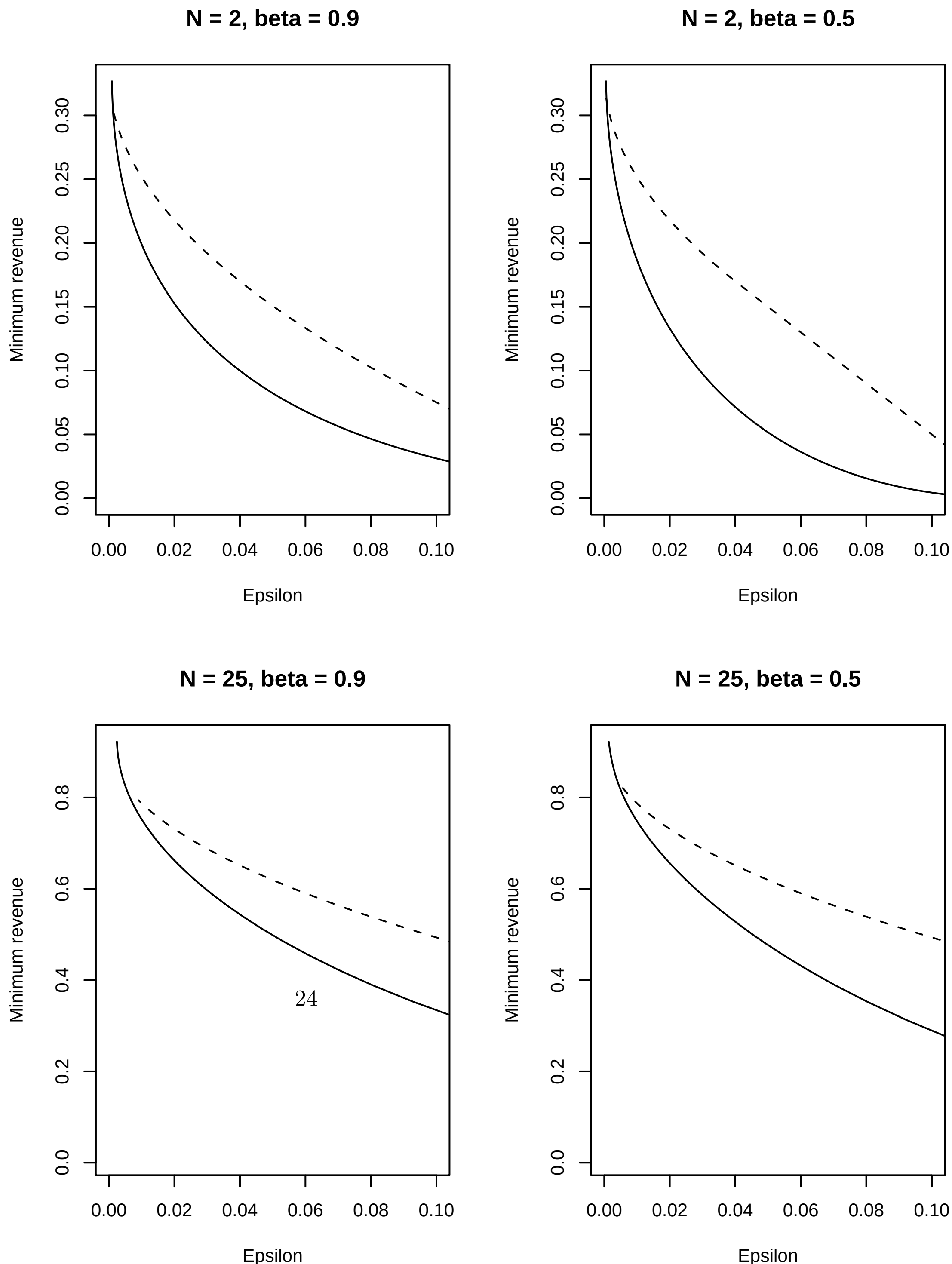
N = 2, beta = 0.9
N = 2, beta = 0.5
N = 25, beta = 0.9
N = 25, beta = 0.5
Minimum revenue
Epsilon
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.0
0.2
0.4
0.6
0.8
0.00
0.02
0.04
0.06
0.08
0.10